

Deflection angle in the strong deflection limit in a general asymptotically flat, static, spherically symmetric spacetime

Naoki Tsukamoto*

School of Physics, Huazhong University of Science and Technology, Wuhan 430074, China

Abstract

Gravitational lensing by the light sphere of compact objects like black holes and wormholes will give us information on the compact objects. In this paper, we provide an improved strong deflection limit analysis in a general asymptotically flat, static, spherically symmetric spacetime. The strong deflection limit analysis also works in superstatic spacetimes. As an example of a superstatic spacetime, we reexamine the deflection angle in the strong deflection limit in an Ellis wormhole spacetime. Using the strong deflection limit analysis, we obtain the deflection angle in the strong deflection limit analytically in the Reissner-Nordström spacetime. The point of the improvement is the definition of a variable z in the strong deflection limit analysis. We show that the choice of the variable z is as important as the choice of the coordinate and we conclude that one should choose a proper variable z for a given spacetime.

* tsukamoto@rikkyo.ac.jp

I. INTRODUCTION

Gravitational lenses are a good tool to search the mass of dark gravitational objects between an observer and a source. Gravitational lenses under a weak-field approximation have been investigated with intensity [1–3] while gravitational lensing without the weak-field approximation has been also studied for a long time [4, 5]. In 1959 Darwin pointed out that faint images appear near a light sphere or a photon sphere [6–8] in the Schwarzschild spacetime [9]. Images lensed by a light sphere have been revived by several authors [6, 10–25] and images near a light sphere of a Reissner-Nordstrom black hole [16, 24, 26, 27], the Kerr black hole [28–31], braneworld black holes [32–34], the other various black holes [35–53], naked singularities [16, 22, 54, 55], and wormholes [17, 55–64] have been investigated.

In [16], Bozza presented a formalism to obtain the deflection angle $\alpha(b)$ of a light in the strong deflection limit $b \rightarrow b_c$, where b is the impact parameter of the light and b_c is the critical impact parameter when the light ray winds around a light sphere, in a general static spherically symmetric spacetime. The deflection angle in the strong deflection limit $b \rightarrow b_c$ is expressed as

$$\alpha(b) = -\bar{a} \log \left(\frac{b}{b_c} - 1 \right) + \bar{b} + O((b - b_c) \log(b - b_c)), \quad (1.1)$$

where \bar{a} is a positive function and \bar{b} is a function.¹

The functions \bar{a} and \bar{b} in the deflection angle in the strong deflection limit are one of fundamental values characterizing a light sphere or the strong-gravitational region of a spacetime. Bozza clearly showed that the positions and the magnifications of images near a light sphere depend on \bar{a} and \bar{b} and the effect of \bar{a} and \bar{b} on the separation and the magnification of images lensed by the light sphere of a supermassive black hole at the center of our galaxy might be measured in near future [16] and then dozens researchers have been discussed them in various spacetimes. (See Ref. [14, 21, 23] for a numerical approach.) Holz and Wheeler have suggested the survey of light curves of light rays which reflected by the light sphere of a black hole near the solar system [25]. The gravitational lensing is called retrolensing and the strong deflection limit analysis is useful to obtain retrolensing light curves [24, 27, 64, 65]. The effect of images appearing near a light sphere on microlensing light curves also has

¹ The order of the error term in the deflection angle presented by Bozza [16] was $O(b - b_c)$. Recently, Tsukamoto pointed out that the error term is underestimated [64].

been discussed in Refs. [53, 64, 66]. Finite-distance corrections to the deflection angle in the strong deflection limit have been investigated in [67]. Relations between the functions \bar{a} and \bar{b} and quasinormal modes [68, 69] and high-energy absorption cross sections [70] have been considered.

Recently, Tsukamoto pointed out that a strong deflection limit analysis presented by Bozza [16] does not work in superstatic or ultrastatic spacetimes and obtained a deflection angle in the strong deflection limit in a superstatic Ellis spacetime in trial-and-error methods [64]. Here we name a spacetime with a time translational Killing vector which has a constant norm ultrastatic or superstatic spacetime. Very recently, Tsukamoto and Gong obtained a deflection angle in the strong deflection limit analytically in Reissner-Nordström spacetime [27] while it cannot be obtained \bar{b} analytically in the deflection angle with the strong deflection limit analysis provided in Ref. [16].

In this paper, we reconsider the strong deflection limit analysis in a general asymptotically flat, static, spherically symmetric spacetime and provide a new formalism working well in superstatic spacetimes. We show also that we can obtain \bar{b} analytically in the deflection angle in the Reissner-Nordström spacetime with the improved strong deflection limit analysis.

This paper is organized as follows. In Sec. II we obtain the formula of a deflection angle in the strong deflection limit in a general asymptotically flat, static, spherically symmetric spacetime. In Sec. III we apply the formula to the Schwarzschild spacetime, the Reissner-Nordström spacetime, and the Ellis wormhole spacetime. In Sec. IV we summarize our result. In this paper we use the units in which the light speed and Newton's constant are unity.

II. DEFLECTION ANGLE IN THE STRONG DEFLECTION LIMIT

In this section, we give an improved method to obtain the deflection angle of a light ray in the strong deflection limit in a general asymptotically flat, static, and spherically symmetric spacetime. The line element is described by

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

where $A(r)$, $B(r)$, and $C(r)$ satisfy an asymptotically-flat condition ²

$$\lim_{r \rightarrow \infty} A(r) = 1, \quad (2.5)$$

$$\lim_{r \rightarrow \infty} B(r) = 1, \quad (2.6)$$

$$\lim_{r \rightarrow \infty} C(r) = r^2. \quad (2.7)$$

There exist time translational and axial Killing vectors $t^\mu \partial_\mu = \partial_t$ and $\phi^\mu \partial_\mu = \partial_\phi$ since the spacetime is static and spherical symmetric, respectively. If a spacetime has a time translational Killing vector with a constant norm, i.e., if $A(r)$ is constant, we name the spacetime superstatic spacetime or ultrastatic spacetime. In any superstatic spacetime, we can transform constant $A(r)$ into unity without loss of generality. Thus, superstatic spacetimes can satisfy the condition (2.5).

We assume that there is at least one positive solution of $D(r) = 0$, where

$$D(r) \equiv \frac{C'(r)}{C(r)} - \frac{A'(r)}{A(r)}, \quad (2.8)$$

where $'$ denotes the differentiation with respect to the radial coordinate r . We call the largest positive solution of $D(r) = 0$ the radius of a light sphere r_m . We assume that $A(r)$, $B(r)$, and $C(r)$ are finite and positive for $r \geq r_m$. ³

The trajectory of a light is described by $g_{\mu\nu} k^\mu k^\nu = 0$, where $k^\mu \equiv \dot{x}^\mu$ is the wave number of the photon and $\dot{}$ denotes the differentiation with respect to an affine parameter parameterizing the trajectory. The conserved energy $E \equiv -g_{\mu\nu} t^\mu k^\nu = A(r)\dot{t}$ and the conserved angular momentum $L \equiv g_{\mu\nu} \phi^\mu k^\nu = C(r)\dot{\phi}$ are constant along it. We assume that E and L do not vanish. We define the impact parameter b as

$$b \equiv \frac{L}{E} = \frac{C(r)\dot{\phi}}{A(r)\dot{t}}. \quad (2.9)$$

Without loss of generality, we can assume $\theta = \pi/2$ because of spherical symmetry. The trajectory equation is expressed as

$$-A(r)\dot{t}^2 + B(r)\dot{r}^2 + C(r)\dot{\phi}^2 = 0 \quad (2.10)$$

² In Ref. [16], an following asymptotically-flat condition is assumed:

$$\lim_{r \rightarrow \infty} A(r) = 1 - \frac{2M}{r}, \quad (2.2)$$

$$\lim_{r \rightarrow \infty} B(r) = 1 + \frac{2M}{r}, \quad (2.3)$$

$$\lim_{r \rightarrow \infty} C(r) = r^2, \quad (2.4)$$

where M is the Arnowitt-Deser-Misner (ADM) mass.

³ In Ref. [16], $A'(r) > 0$ and $C'(r) > 0$ for $r > r_m$ are also assumed. We extend a formalism presented by Bozza [16] to obtain a deflection angle in the strong deflection limit in superstatic spacetimes with $A'(r) = 0$ everywhere.

or

$$\dot{r}^2 = V(r), \quad (2.11)$$

where the effective potential $V(r)$ for the motion of a photon is defined as

$$V(r) \equiv \frac{L^2 R(r)}{B(r)C(r)}, \quad (2.12)$$

where $R(r)$ is given by

$$R(r) \equiv \frac{C(r)}{A(r)b^2} - 1. \quad (2.13)$$

The motion of the photon is permitted in the region $V(r) \geq 0$. Since we obtain $\lim_{r \rightarrow \infty} V(r) = E^2 > 0$ from the asymptotically-flat condition (2.5), the photon can exist at infinity $r \rightarrow \infty$. We assume that $R(r) = 0$ has at least one positive solution.

We consider that a photon approaches a gravitational object from infinity, is scattered at a closest distance $r = r_0$, and goes infinity. In the scatter case, $r_m < r_0$ should be satisfied. Please note that $r = r_0$ is the largest positive solution of $R(r) = 0$ and that $B(r)$ and $C(r)$ do not diverge for $r \geq r_0$. Thus, $V(r)$ vanishes at $r = r_0$. Since \dot{r} vanishes at the closest distance $r = r_0$, from the trajectory equation (2.10), we obtain

$$A_0 \dot{t}_0^2 = C_0 \dot{\phi}_0^2. \quad (2.14)$$

Here and hereafter subscript 0 denotes the quantities at $r = r_0$. Without loss of generality, we can assume that the impact parameter b is positive as long as we consider only one light ray. Since the impact parameter is constant along the trajectory, using Eq. (2.14), the impact parameter (2.9) can be expressed as

$$b(r_0) = \frac{L}{E} = \frac{C_0 \dot{\phi}_0}{A_0 \dot{t}_0} = \sqrt{\frac{C_0}{A_0}}. \quad (2.15)$$

Using Eq. (2.15), we can rewrite $R(r)$ into

$$R(r) = \frac{A_0 C}{A C_0} - 1. \quad (2.16)$$

We show a necessary and sufficient condition for existing of a circular light orbit by following Hasse and Perlick [8]. We express the trajectory equation as

$$\frac{BC\dot{r}^2}{E^2} + b^2 = \frac{C}{A}. \quad (2.17)$$

Differentiating Eq. (2.17) with respect to the affine parameter and then dividing it by \dot{r} , we obtain

$$\ddot{r} + \frac{1}{2} \left(\frac{B'}{B} + \frac{C'}{C} \right) \dot{r}^2 = \frac{E^2}{AB} D(r). \quad (2.18)$$

Since $A(r)$, $B(r)$, and $C(r)$ are finite and positive for $r \geq r_m$ and E is positive, a circular light orbit exists if and only if

$$D(r) = 0. \quad (2.19)$$

Please note that $R'_m = D_m C_m A_m / b^2 = 0$, where subscript m denotes the quantities at $r = r_m$.

We define the critical impact parameter b_c as

$$b_c(r_m) \equiv \lim_{r_0 \rightarrow r_m} \sqrt{\frac{C_0}{A_0}}. \quad (2.20)$$

and name a limit $r_0 \rightarrow r_m$ or $b \rightarrow b_c$ strong deflection limit. The derivative of the effective potential $V(r)$ with respect to r is given by

$$V'(r) = \frac{L^2}{BC} \left[R' + \left(\frac{C'}{C} - \frac{B'}{B} \right) R \right], \quad (2.21)$$

and, hence, we obtain

$$\lim_{r_0 \rightarrow r_m} V(r_0) = \lim_{r_0 \rightarrow r_m} V'(r_0) = 0 \quad (2.22)$$

in the strong deflection limit $r_0 \rightarrow r_m$. This means that the light ray winds around the light sphere in the strong deflection limit.

The trajectory equation of a light is rewritten as

$$\left(\frac{dr}{d\phi} \right)^2 = \frac{R(r)C(r)}{B(r)}. \quad (2.23)$$

and the deflection angle $\alpha(r_0)$ of the light is obtained as

$$\alpha(r_0) = I(r_0) - \pi, \quad (2.24)$$

where $I(r_0)$ is defined as

$$I(r_0) \equiv 2 \int_{r_0}^{\infty} \frac{dr}{\sqrt{\frac{R(r)C(r)}{B(r)}}}. \quad (2.25)$$

Introducing a variable z defined as ⁴

$$z \equiv 1 - \frac{r_0}{r}, \quad (2.27)$$

⁴ In Ref. [16], a counterpart $z_{[16]}$ of the variable z is defined as

$$z_{[16]} \equiv \frac{A - A_0}{1 - A_0}. \quad (2.26)$$

We discuss the details of z and $z_{[16]}$ in Secs. III and IV.

$I(r_0)$ is described by

$$I(r_0) = \int_0^1 f(z, r_0) dz, \quad (2.28)$$

where $f(z, r_0)$ is defined as

$$f(z, r_0) \equiv \frac{2r_0}{\sqrt{G(z, r_0)}}, \quad (2.29)$$

where $G(z, r_0)$ is given by

$$G(z, r_0) \equiv R \frac{C}{B} (1 - z)^4. \quad (2.30)$$

Since the expansions of a function $F(r)$ and its inverse $1/F(r)$ in the power of z are given by

$$F = F_0 + F'_0 r_0 z + \left(\frac{1}{2} F''_0 r_0^2 + F'_0 r_0 \right) z^2 + O(z^3) \quad (2.31)$$

and

$$\begin{aligned} \frac{1}{F} &= \frac{1}{F_0} - \frac{F'_0 r_0}{F_0^2} z \\ &+ \frac{r_0}{F_0^3} \left(-\frac{1}{2} F_0 F''_0 r_0 + F_0'^2 r_0 - F_0 F'_0 \right) z^2 + O(z^3), \end{aligned} \quad (2.32)$$

respectively, $R(r)$ can be expanded in the power of z as

$$\begin{aligned} R(r) &= D_0 r_0 z + \left[\frac{r_0}{2} \left(\frac{C''_0}{C_0} - \frac{A''_0}{A_0} \right) \right. \\ &\quad \left. + \left(1 - \frac{A'_0 r_0}{A_0} \right) D_0 \right] r_0 z^2 + O(z^3). \end{aligned} \quad (2.33)$$

Using Eqs. (2.30)-(2.33), we obtain the expansion of $G(z, r_0)$ in the power of z as

$$G(z, r_0) = \sum_{n=1}^{\infty} c_n(r_0) z^n, \quad (2.34)$$

where $c_1(r_0)$ and $c_2(r_0)$ are given by

$$c_1(r_0) = \frac{C_0 D_0 r_0}{B_0} \quad (2.35)$$

and

$$\begin{aligned} c_2(r_0) &= \frac{C_0 r_0}{B_0} \left\{ D_0 \left[\left(D_0 - \frac{B'_0}{B_0} \right) r_0 - 3 \right] \right. \\ &\quad \left. + \frac{r_0}{2} \left(\frac{C''_0}{C_0} - \frac{A''_0}{A_0} \right) \right\}, \end{aligned} \quad (2.36)$$

respectively.

In the strong deflection limit $r_0 \rightarrow r_m$, from $D_m = 0$, we obtain

$$c_1(r_m) = 0 \quad (2.37)$$

and

$$c_2(r_m) = \frac{C_m r_m^2}{2B_m} D'_m, \quad (2.38)$$

where

$$D'_m = \frac{C''_m}{C_m} - \frac{A''_m}{A_m} \quad (2.39)$$

and, hence, we obtain

$$G_m(z) = c_2(r_m)z^2 + O(z^3). \quad (2.40)$$

This shows that the leading order of the divergence of $f(z, r_0)$ is z^{-1} and that the integral $I(r_0)$ diverges logarithmically in the strong deflection limit $r_0 \rightarrow r_m$.

We separate the integral $I(r_0)$ into a divergent part $I_D(r_0)$ and a regular part $I_R(r_0)$. We define the divergent part $I_D(r_0)$ as

$$I_D(r_0) \equiv \int_0^1 f_D(z, r_0) dz, \quad (2.41)$$

where $f_D(z, r_0)$ is defined by

$$f_D(z, r_0) \equiv \frac{2r_0}{\sqrt{c_1(r_0)z + c_2(r_0)z^2}}. \quad (2.42)$$

We can integrate $I_D(r_0)$ and obtain

$$I_D(r_0) = \frac{4r_0}{\sqrt{c_2(r_0)}} \log \frac{\sqrt{c_2(r_0)} + \sqrt{c_1(r_0) + c_2(r_0)}}{\sqrt{c_1(r_0)}}. \quad (2.43)$$

Since the expansions of $c_1(r_0)$ and $b(r_0)$ in power of $r_0 - r_m$ are given by

$$c_1(r_0) = \frac{C_m r_m D'_m}{B_m} (r_0 - r_m) + O((r_0 - r_m)^2) \quad (2.44)$$

and

$$b(r_0) = b_c(r_m) + \frac{1}{4} \sqrt{\frac{C_m}{A_m}} D'_m (r_0 - r_m)^2 + O((r_0 - r_m)^3), \quad (2.45)$$

respectively, $c_1(r_0)$ in the strong deflection limit $r_0 \rightarrow r_m$ is described by

$$\lim_{r_0 \rightarrow r_m} c_1(r_0) = \lim_{b \rightarrow b_c} \frac{2C_m r_m \sqrt{D'_m}}{B_m} \left(\frac{b}{b_c} - 1 \right)^{\frac{1}{2}}. \quad (2.46)$$

Thus, we obtain the divergent part $I_D(b)$ in the strong deflection limit $b \rightarrow b_c$ as

$$I_D(b) = -\frac{r_m}{\sqrt{c_2(r_m)}} \log\left(\frac{b}{b_c} - 1\right) + \frac{r_m}{\sqrt{c_2(r_m)}} \log r_m^2 D'_m + O((b - b_c) \log(b - b_c)). \quad (2.47)$$

We define the regular part $I_R(r_0)$ as

$$I_R(r_0) \equiv \int_0^1 f_R(z, r_0) dz, \quad (2.48)$$

where $f_R(r_0)$ is defined by

$$f_R(r_0) \equiv f(z, r_0) - f_D(z, r_0). \quad (2.49)$$

Please notice $I(r_0) = I_D(r_0) + I_R(r_0)$. We expand $I_R(r_0)$ in power of $r_0 - r_m$ and we concentrate on the leading term $I_R(r_m)$ since we are interested in the regular part I_R in the strong deflection limit $r_0 \rightarrow r_m$ or $b \rightarrow b_c$. We should integrate analytically or numerically the regular part

$$I_R(r_0) = \int_0^1 f_R(z, r_m) dz + O((r_0 - r_m) \log(r_0 - r_m)) \quad (2.50)$$

or

$$I_R(b) = \int_0^1 f_R(z, b_c) dz + O((b - b_c) \log(b - b_c)). \quad (2.51)$$

The deflection angle in the strong deflection limit $r_0 \rightarrow r_m$ or $b \rightarrow b_c$ is given by

$$\alpha(b) = -\bar{a} \log\left(\frac{b}{b_c} - 1\right) + \bar{b} + O((b - b_c) \log(b - b_c)), \quad (2.52)$$

where \bar{a} and \bar{b} are given by

$$\bar{a} = \sqrt{\frac{2B_m A_m}{C_m'' A_m - C_m A_m''}} \quad (2.53)$$

and

$$\bar{b} = \bar{a} \log\left[r_m^2 \left(\frac{C_m''}{C_m} - \frac{A_m''}{A_m}\right)\right] + I_R(r_m) - \pi, \quad (2.54)$$

respectively.

III. APPLICATIONS

In this section, we apply our result obtained in the previous section for the Schwarzschild spacetime, the Reissner-Nordström spacetime, and an Ellis wormhole spacetime.

A. Schwarzschild black hole

The line element in the Schwarzschild spacetime is given by

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.1)$$

and then

$$A(r) = 1 - \frac{2M}{r}, \quad (3.2)$$

$$B(r) = \frac{1}{1 - \frac{2M}{r}}, \quad (3.3)$$

$$C(r) = r^2. \quad (3.4)$$

From Eq. (2.8), we obtain $r_m = 3M$. There is an event horizon at $r = r_H \equiv 2M$ and $B(r)$ diverges there but all the assumptions are satisfied because of $r_m > r_H$.

From Eq. (2.20), the critical impact parameter b_c is given by

$$b_c = 3\sqrt{3}M. \quad (3.5)$$

From Eqs. (2.53) and (2.54) we obtain

$$\bar{a} = 1 \quad (3.6)$$

and

$$\bar{b} = \log 6 + I_R(r_m) - \pi, \quad (3.7)$$

respectively.

From Eq. (2.50), we obtain $I_R(r_m)$ as

$$\begin{aligned} I_R(r_m) &= 2 \int_0^1 \left(\frac{1}{z\sqrt{1 - \frac{2}{3}z}} - \frac{1}{z} \right) dz \\ &= 2 \log[6(2 - \sqrt{3})]. \end{aligned} \quad (3.8)$$

and, hence, we obtain \bar{b} as

$$\bar{b} = \log \lambda - \pi, \quad (3.9)$$

where $\lambda \equiv 216(7 - 4\sqrt{3})$. This is the same result obtained in Ref. [16].

In Ref. [16], Bozza used a counterpart $z_{[16]}$ of the variable z is defined as

$$z_{[16]} \equiv \frac{A - A_0}{1 - A_0}. \quad (3.10)$$

We realize that the formalism presented in Sec II to obtain the deflection angle in the strong deflection angle is the same as one in [16] in the Schwarzschild spacetime since z is equivalent to $z_{[16]}$:

$$z = z_{[16]} = 1 - \frac{r_0}{r}. \quad (3.11)$$

Iyer and Petters [18] expanded the deflection angle in the Schwarzschild spacetime as an affine perturbation series

$$\begin{aligned} \alpha = & -\log\left(\frac{b}{b_c} - 1\right) + \log\lambda - \pi \\ & + \sum_{n=1}^{\infty} \left(\frac{b}{b_c} - 1\right)^n \left\{ \rho_n - \sigma_n \log\left[\left(\frac{b}{b_c} - 1\right) \frac{1}{\lambda}\right] \right\}, \end{aligned} \quad (3.12)$$

where ρ_n and σ_n are constant. They showed ρ_n and σ_n clearly for $n \leq 3$. This fact shows that the order of the error term of the deflection angle in the strong deflection limit is not $O(b_c - b)$ but $O((b_c - b) \log(b_c - b))$.

B. Reissner-Nordström black hole

Very recently, Tsukamoto and Gong obtained the deflection angle in the strong deflection limit analytically in the Reissner-Nordström spacetime [27]. They used the variable z defined as Eq. (2.27). We apply the formula of the deflection angle in the strong deflection limit presented in Section II for the Reissner-Nordström spacetime to test the formula.

The line element of the Reissner-Nordström spacetime is given by

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \\ & + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \quad (3.13)$$

and $A(r)$, $B(r)$, and $C(r)$ are given by

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (3.14)$$

$$B(r) = \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}, \quad (3.15)$$

$$C(r) = r^2. \quad (3.16)$$

If the electrical charge Q satisfies $Q \leq M$, there is an event horizon at $r = r_H \equiv M + \sqrt{M^2 - Q^2}$. We concentrate on the black hole spacetime with $Q \leq M$. From Eq. (2.8),

There is a light sphere at $r = r_m$, where r_m is given by

$$r_m = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2}. \quad (3.17)$$

Note that $B(r)$ is finite and positive in the range $r \geq r_m$ since $r_H < r_m$.

From Eq. (2.20), the critical impact parameter b_c is given by

$$b_c = \frac{r_m^2}{\sqrt{Mr_m - Q^2}}. \quad (3.18)$$

We obtain \bar{a} and \bar{b} as, from Eqs. (2.53) and (2.54),

$$\bar{a} = \frac{r_m}{\sqrt{3Mr_m - 4Q^2}} \quad (3.19)$$

and

$$\bar{b} = \bar{a} \log \frac{2(3Mr_m - 4Q^2)}{Mr_m - Q^2} + I_R(r_m) - \pi, \quad (3.20)$$

respectively. From Eq. (2.50), the regular integral is given by

$$\begin{aligned} I_R(r_m) &= \int_0^1 \frac{2r_m dz}{z \sqrt{3Mr_m - 4Q^2 - 2(Mr_m - 2Q^2)z - Q^2 z^2}} \\ &\quad - \int_0^1 \frac{2\bar{a}}{z} dz. \\ &= \bar{a} \log \left[\frac{4(3Mr_m - 4Q^2)^2}{M^2 r_m^2 (Mr_m - Q^2)} \right. \\ &\quad \left. \times \left(2\sqrt{Mr_m - Q^2} - \sqrt{3Mr_m - 4Q^2} \right)^2 \right] \end{aligned} \quad (3.21)$$

Thus, we obtain \bar{b} as

$$\begin{aligned} \bar{b} &= \bar{a} \log \left[\frac{8(3Mr_m - 4Q^2)^3}{M^2 r_m^2 (Mr_m - Q^2)^2} \right. \\ &\quad \left. \times \left(2\sqrt{Mr_m - Q^2} - \sqrt{3Mr_m - 4Q^2} \right)^2 \right] - \pi. \end{aligned} \quad (3.22)$$

When $Q = 0$, we obtain b_c , \bar{a} , and \bar{b} which are equal to b_c , \bar{a} , and \bar{b} in the Schwarzschild spacetime. When the maximal charged black hole case ($Q = M$), b_c , \bar{a} , and \bar{b} become

$$b_c = 4M, \quad (3.23)$$

$$\bar{a} = \sqrt{2}, \quad (3.24)$$

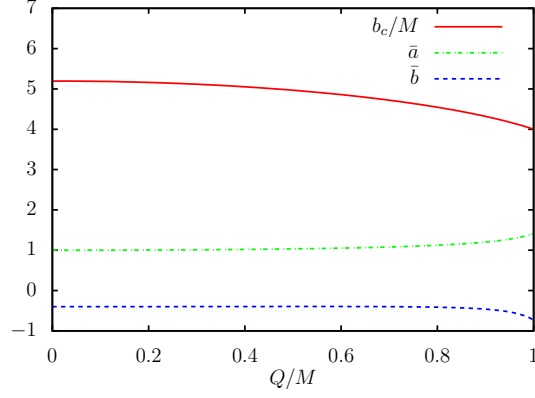


FIG. 1. b_c/M , \bar{a} , and \bar{b} in the Reissner-Nordström black hole spacetime. The solid (red), dot-dashed (green), and dashed (blue) curves denote b_c/M , \bar{a} , and \bar{b} , respectively.

and

$$\bar{b} = 2\sqrt{2}\log[4(2 - \sqrt{2})] - \pi, \quad (3.25)$$

respectively. Figure 1 shows b_c/M , \bar{a} , and \bar{b} as a function of Q/M . These are the same ones obtained in Ref. [27].

In Ref. [16], Bozza calculated numerically the counterpart of the regular integral $I_R(r_m)$ but could not calculate it analytically without expanding it in power of $(Q/M)^2$. We notice that the variable $z_{[16]}$ in the Reissner-Nordström spacetime becomes

$$z_{[16]} = 1 - \frac{r_0^2(2Mr - Q^2)}{r^2(2Mr_0 - Q^2)}. \quad (3.26)$$

The variable $z_{[16]}$ seems not to be suitable in the Reissner-Nordström spacetime.

C. Ellis wormhole

An Ellis wormhole [71, 72] is one of the simplest Morris-Thorne wormhole solution [73]. Gravitational Lensing by the Ellis wormhole was investigated [17, 55–61, 74–86] and the upper bound of the number density [74, 75] was given from surveys of gravitational lensing. The line element in the Ellis wormhole spacetime is given by

$$ds^2 = -dt^2 + dl^2 + (l^2 + a^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.27)$$

where l is a radial coordinate defined in the range $-\infty < l < \infty$ and a is a positive constant. There is a throat at $l = 0$. We concentrate on a light ray which does not pass the throat

and which exists in a region $l \geq 0$. The Ellis wormhole has vanishing ADM masses [64]. It is a superstatic spacetime since there is a time translational Killing vector with a constant norm. By solving an equation

$$g_{tt}(l) \frac{dg_{\theta\theta}(l)}{dl} - g_{\theta\theta}(l) \frac{dg_{tt}(l)}{dl} = 0, \quad (3.28)$$

we find a light sphere at $l = 0$. Since we have assume a positive radius of a light sphere in the previous section, we introduce a new radius coordinate $r \equiv l + p$, where p is a positive constant. The line element is rewritten as

$$ds^2 = -dt^2 + dr^2 + [(r - p)^2 + a^2](d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.29)$$

and we obtain

$$A(r) = 1, \quad (3.30)$$

$$B(r) = 1, \quad (3.31)$$

$$C(r) = (r - p)^2 + a^2. \quad (3.32)$$

Under the radial coordinate r , the radius of the light sphere $r_m = p$ is positive. The light sphere at $r = r_m$ is coincident with the wormhole throat. Please note that one cannot apply the strong deflection limit analysis in Ref. [16] for any superstatic spacetime because of the violation of an assumption that $A'(r)$ is positive for $r > r_m$ while one can apply our result investigated in Sec. II for the Ellis wormhole spacetime with the line element (3.29). In general, we cannot define the variable $z_{[16]}$ in a superstatic spacetime satisfying the asymptotically-flat condition (2.5).

From Eq. (2.20), we obtain the critical impact parameter b_c as

$$b_c = a. \quad (3.33)$$

Using Eqs. (2.53) and (2.54), we obtain \bar{a} and \bar{b} as

$$\bar{a} = 1 \quad (3.34)$$

and

$$\bar{b} = \log \frac{2p^2}{a^2} + I_R(r_m) - \pi, \quad (3.35)$$

respectively. From Eq. (2.50), we can obtain the regular integral I_R as

$$\begin{aligned} I_R(r_m) &= 2 \int_0^1 \left(\frac{a}{z \sqrt{a^2 - 2a^2 z + (a^2 + p^2) z^2}} - \frac{1}{z} \right) dz \\ &= 2 \log \frac{2a}{p}. \end{aligned} \quad (3.36)$$

and hence we obtain

$$\bar{b} = 3 \log 2 - \pi. \quad (3.37)$$

The deflection angle is equal to the deflection angle in the strong deflection limit obtained in Ref. [64]. Tsukamoto [64] used a variable $z_{[64]}$ defined as

$$z_{[64]} \equiv 1 - \frac{b}{\sqrt{r^2 + a^2}} = 1 - \frac{\sqrt{r_0^2 + a^2}}{\sqrt{r^2 + a^2}}. \quad (3.38)$$

Since $z_{[64]}$ is not equivalent to $z = 1 - r_0/r$ (2.27) and $z_{[16]}$ in the Ellis wormhole spacetime, the calculation which we have presented here can be an additional crosscheck for the deflection angle in the strong deflection limit in the Ellis wormhole spacetime. See [64] for the details of the deflection angle in the Ellis wormhole spacetime.

IV. CONCLUSION

In this paper, we have investigated a strong deflection limit analysis in a general asymptotically flat, static, spherically symmetric spacetime. The improved strong deflection limit analysis works in superstatic spacetimes while the one presented in Ref. [16] does not. As an example of a superstatic spacetime, we have investigated the deflection angle in the strong deflection limit in an Ellis wormhole spacetime. Using the improved strong deflection limit analysis, we have obtained the deflection angle in the strong deflection limit analytically in the Reissner-Nordström spacetime while it cannot be obtained in Ref. [16]. The most important point of the improvement of the strong deflection limit analysis is the definition of a variable z (2.27). In this paper, we have chosen a simple variable z not only in the Schwarzschild spacetime but also in the other spacetimes. We never insist on that we have chosen the best definition of the variable z in this paper but we show clearly that the choice of the variable z is as important as the choice of the coordinate. We conclude that one should choose a proper variable z for a given spacetime. Even if the variable z (2.27) is not suitable in a specific spacetime, our strong deflection limit analysis in this paper will give us a clue to find a more proper variable z in the spacetime.

ACKNOWLEDGEMENTS

The author thanks Ken-ichi Nakao, Tetsuya Shiromizu, Chul-Moon Yoo, Takahisa Igata, and Yungui Gong for valuable comments. He also thanks Tomohiro Harada, Yoshimune Tomikawa, Hideki Asada, Hirotaka Yoshino, Yusuke Suzuki, Rio Saitou, and Takafumi Kokubu for useful conversations. This research was supported in part by the Natural Science Foundation of China under Grant No. 11475065 and the Program for New Century Excellent Talents in University under Grant No. NCET-12-0205.

-
- [1] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
 - [2] A. O. Petters, H. Levine, and J. Wambsganss, *Singularity Theory and Gravitational Lensing* (Birkhauser, Boston, 2001).
 - [3] P. Schneider, C. S. Kochanek, and J. Wambsganss, *Gravitational Lensing: Strong, Weak and Micro, Lecture Notes of the 33rd Saas-Fee Advanced Course*, edited by G. Meylan, P. Jetzer, and P. North (Springer-Verlag, Berlin, 2006).
 - [4] V. Perlick, *Living Rev. Relativity* **7**, 9 (2004).
 - [5] V. Bozza, *Gen. Relativ. Gravit.* **42**, 2269 (2010).
 - [6] R. d' E. Atkinson, *Astron. J.* **70**, 517 (1965).
 - [7] C. M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, *J. Math. Phys.* **42**, 818 (2001).
 - [8] W. Hasse and V. Perlick, *Gen. Relativ. Gravit.* **34**, 415 (2002).
 - [9] C. Darwin, *Proc. R. Soc. Lond. A* **249** (1959).
 - [10] J.-P. Luminet, *Astron. Astrophys.* **75**, 228 (1979).
 - [11] H. C. Ohanian, *Am. J. Phys.* **55**, 428 (1987).
 - [12] R. J. Nemiroff, *Am. J. Phys.* **61**, 619 (1993).
 - [13] S. Frittelli, T. P. Kling, and E. T. Newman, *Phys. Rev. D* **61**, 064021 (2000).
 - [14] K. S. Virbhadra and G. F. R. Ellis, *Phys. Rev. D* **62**, 084003 (2000).
 - [15] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, *Gen. Relativ. Gravit.* **33**, 1535 (2001).
 - [16] V. Bozza, *Phys. Rev. D* **66**, 103001 (2002).
 - [17] V. Perlick, *Phys. Rev. D* **69**, 064017 (2004).
 - [18] S. V. Iyer and A. O. Petters, *Gen. Rel. Grav.* **39**, 1563 (2007).

- [19] V. Bozza and M. Sereno, Phys. Rev. D **73**, 103004 (2006).
- [20] V. Bozza and G. Scarpetta, Phys. Rev. D **76**, 083008 (2007).
- [21] V. Bozza, Phys. Rev. D **78**, 103005 (2008).
- [22] K. S. Virbhadra and C. R. Keeton, Phys. Rev. D **77**, 124014 (2008).
- [23] K. S. Virbhadra, Phys. Rev. D **79**, 083004 (2009).
- [24] E. F. Eiroa and D. F. Torres, Phys. Rev. D **69**, 063004 (2004).
- [25] D. E. Holz and J. A. Wheeler, Astrophys. J. **578**, 330 (2002).
- [26] E. F. Eiroa, G. E. Romero, and D. F. Torres, Phys. Rev. D **66**, 024010 (2002).
- [27] N. Tsukamoto and Y. Gong, *Preprint* (2016).
- [28] V. Bozza, Phys. Rev. D **67**, 103006 (2003).
- [29] V. Bozza, F. De Luca, and G. Scarpetta, Phys. Rev. D **74**, 063001 (2006).
- [30] V. Bozza, F. De Luca, G. Scarpetta, and M. Sereno, Phys. Rev. D **72**, 083003 (2005).
- [31] H. Saida, arXiv:1606.04716 [astro-ph.HE].
- [32] E. F. Eiroa, Phys. Rev. D **71**, 083010 (2005).
- [33] R. Whisker, Phys. Rev. D **71**, 064004 (2005).
- [34] E. F. Eiroa and C. M. Sendra, Phys. Rev. D **86**, 083009 (2012).
- [35] A. Bhadra, Phys. Rev. D **67**, 103009 (2003).
- [36] E. F. Eiroa, Phys. Rev. D **73**, 043002 (2006).
- [37] N. Mukherjee and A. S. Majumdar, Gen. Rel. Grav. **39**, 583 (2007).
- [38] G. N. Gyulchev and S. S. Yazadjiev, Phys. Rev. D **75**, 023006 (2007).
- [39] S. b. Chen and J. l. Jing, Phys. Rev. D **80**, 024036 (2009).
- [40] Y. Liu, S. Chen, and J. Jing, Phys. Rev. D **81**, 124017 (2010).
- [41] E. F. Eiroa and C. M. Sendra, Class. Quant. Grav. **28**, 085008 (2011).
- [42] C. Ding, S. Kang, C. Y. Chen, S. Chen, and J. Jing, Phys. Rev. D **83**, 084005 (2011).
- [43] S. Chen, Y. Liu, and J. Jing, Phys. Rev. D **83**, 124019 (2011).
- [44] S. W. Wei, Y. X. Liu, C. E. Fu, and K. Yang, JCAP **1210**, 053 (2012).
- [45] M. Azreg-Ainou, Phys. Rev. D **87**, 024012 (2013).
- [46] G. N. Gyulchev and I. Z. Stefanov, Phys. Rev. D **87**, 063005 (2013).
- [47] E. F. Eiroa and C. M. Sendra, Phys. Rev. D **88**, 103007 (2013).
- [48] S. W. Wei, K. Yang, and Y. X. Liu, Eur. Phys. J. C **75**, 253 (2015) Erratum: [Eur. Phys. J. C **75**, 331 (2015)].

- [49] E. F. Eiroa and C. M. Sendra, *Eur. Phys. J. C* **74**, 3171 (2014).
- [50] S. Sahu, K. Lochan, and D. Narasimha, *Phys. Rev. D* **91**, 063001 (2015).
- [51] H. Sotani and U. Miyamoto, *Phys. Rev. D* **92**, 044052 (2015).
- [52] S. S. Zhao and Y. Xie, *JCAP* **1607**, 007 (2016).
- [53] N. Tsukamoto, T. Kitamura, K. Nakajima, and H. Asada, *Phys. Rev. D* **90**, 064043 (2014).
- [54] K. S. Virbhadra and G. F. R. Ellis, *Phys. Rev. D* **65**, 103004 (2002).
- [55] T. K. Dey and S. Sen, *Mod. Phys. Lett. A*, **23**, 953 (2008).
- [56] L. Chetouani and G. Clément, *Gen. Relativ. Gravit.* **16**, 111 (1984).
- [57] K. K. Nandi, Y. Z. Zhang, and A. V. Zakharov, *Phys. Rev. D* **74**, 024020 (2006).
- [58] A. Bhattacharya and A. A. Potapov, *Mod. Phys. Lett. A* **25**, 2399 (2010).
- [59] G. W. Gibbons and M. Vyska, *Class. Quant. Grav.* **29**, 065016 (2012).
- [60] K. Nakajima and H. Asada, *Phys. Rev. D* **85**, 107501 (2012).
- [61] N. Tsukamoto, T. Harada, and K. Yajima, *Phys. Rev. D* **86**, 104062 (2012).
- [62] K. K. Nandi, A. A. Potapov, R. N. Izmailov, A. Tamang, and J. C. Evans, *Phys. Rev. D* **93**, 104044 (2016).
- [63] N. Tsukamoto and T. Harada, *arXiv:1607.01120 [gr-qc]*.
- [64] N. Tsukamoto, *Phys. Rev. D* **94**, 124001 (2016).
- [65] V. Bozza and L. Mancini, *Astrophys. J.* **611**, 1045 (2004).
- [66] A. O. Petters, *Mon. Not. Roy. Astron. Soc.* **338**, 457 (2003).
- [67] A. Ishihara, Y. Suzuki, T. Ono, and H. Asada, *arXiv:1612.04044 [gr-qc]*.
- [68] I. Z. Stefanov, S. S. Yazadjiev, and G. G. Gyulchev, *Phys. Rev. Lett.* **104**, 251103 (2010).
- [69] S. W. Wei and Y. X. Liu, *Phys. Rev. D* **89**, 047502 (2014).
- [70] S. W. Wei, Y. X. Liu, and H. Guo, *Phys. Rev. D* **84**, 041501 (2011).
- [71] H. G. Ellis, *J. Math. Phys.* **14**, 104 (1973).
- [72] K. A. Bronnikov, *Acta Phys. Pol. B* **4**, 251 (1973).
- [73] M. S. Morris and K. S. Thorne, *Am. J. Phys.* **56**, 395 (1988).
- [74] R. Takahashi and H. Asada, *Astrophys. J.* **768**, L16 (2013).
- [75] C. M. Yoo, T. Harada, and N. Tsukamoto, *Phys. Rev. D* **87**, 084045 (2013).
- [76] T. Muller, *Phys. Rev. D* **77**, 044043 (2008).
- [77] F. Abe, *Astrophys. J.* **725**, 787 (2010).
- [78] Y. Toki, T. Kitamura, H. Asada, and F. Abe, *Astrophys. J.* **740**, 121 (2011).

- [79] N. Tsukamoto and T. Harada, Phys. Rev. D **87**, 024024 (2013).
- [80] T. Kitamura, K. Nakajima, and H. Asada, Phys. Rev. D **87**, 027501 (2013).
- [81] K. Izumi, C. Hagiwara, K. Nakajima, T. Kitamura, and H. Asada, Phys. Rev. D **88**, 024049 (2013).
- [82] T. Kitamura, K. Izumi, K. Nakajima, C. Hagiwara, and H. Asada, Phys. Rev. D **89**, 084020 (2014).
- [83] K. Nakajima, K. Izumi, and H. Asada, Phys. Rev. D **90**, 084026 (2014).
- [84] V. Bozza and A. Postiglione, JCAP **1506**, 036 (2015).
- [85] V. Bozza and C. Melchiorre, JCAP **1603**, 040 (2016).
- [86] R. Lukmanova, A. Kulbakova, R. Izmailov, and A. A. Potapov, Int. J. Theor. Phys. **55**, 4723 (2016).